First-Excursion Failure of Randomly Excited Structures

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The probability for a randomly excited structure to survive a service time interval without suffering a first-excursion failure is determined analytically. The first-excursion failure occurs when, for the first time, the structural response passes out of a prescribed safety domain. The problem is formulated from the viewpoint of the Stratonovich-Kuznetsov theory of random points. The exact solution is expressed in two equivalent series forms, one reducible to Rice's "in and exclusion" series. The first order truncation of the second series corresponds to Poisson random points and the second order truncation to random points with "pseudo" Gaussian arrival rate. Numerical results are presented for a single-degree-of-freedom linear oscillator under Gaussian white noise excitation based on these truncations and the model of nonapproaching random points suggested by Stratonovich.

Introduction

IN the design process of a structure under random noise excitation it is often specified that within a given operational time interval the probability for the response of the structure (be it the deflection or the stress at a critical point) to pass out of prescribed safety bounds must be small. When a safety bound is violated the structure is said to have suffered an excursion failure (as opposed to, for example, fatigue or corrosion failures). From an engineering standpoint we are usually concerned only with the time when the first-excursion failure occurs.

Specifically, let $X(\tau)$ be the response in question. A first-excursion failure is the occurrence, for the first time, of either the event $\{X(\tau) \leq -a\}$ or the event $\{X(\tau) \geq b\}$, where $-a$ and b are the prescribed safety bounds. If the structure's resistance to a first-excursion failure does not change with time, then a and b are two positive constants. It is indicative to express the probability of a first-excursion failure within the time interval $0 < \tau \leq t$ as follows:

$$P(t, -a,b) = \text{Prob}\left[\left\{ \inf_{0<\tau \leq t} X(\tau) \leq -a\right\} \cup \left\{ \sup_{0<\tau \geq t} X(\tau) \geq b\right\}\right] \quad (1)$$

where $\cup$ signifies the union of two sets, and inf and sup denote a greatest lower bound and a least upper bound, respectively. Thus, an acceptable design is one for which the probability $P(t,-a,b)$ is sufficiently small for a given t.

The present paper is concerned with a structural response $X(\tau)$, that is related to a random excitation $F(\tau)$ through a linear differential equation of second order in τ. The exact first-excursion probability in this case is unknown. It is of interest to review briefly the following approximate solutions in the literature. If only the mean function and the correlation function of $F(\tau)$ are specified so that the mean and the mean-square values of $X(\tau)$ and its derivative $(d/dt)X(\tau) = \dot{X}(\tau)$ can be computed, then an upper bound of the probability of the first-excursion failure is obtainable from the generalized Chebyshev inequality.[1] However, this upper bound is usually too crude to be of any practical significance. Shinozuka[2] has shown how a better upper bound as well as a lower bound can be computed if other appropriate information about $X(\tau)$ and $\dot{X}(\tau)$ is known. The specific information required is the joint probability density of $X(\tau)$ and $\dot{X}(\tau)$ and that of $X(\tau_1)$ and $X(\tau_2)$. Shinozuka's bounds are useful to treat the nonstationary case where the excitation is intense only within a short duration. These bounds are again too crude for most other cases. In particular, when the response is stationary these bounds are far apart except in the very beginning of an operational interval; therefore, these bounds provide little useful information for the design of a structure exposed to a stationary noise environment for a reasonable length of time. Crandall, Chandiramani, and Cook[3] obtained some first-excursion probabilities from two numerical procedures for white noise excited single-degree-of-freedom systems. One procedure is to simulate a large number of sample functions of the system response on a digital computer, and to compute the sample distribution of the random time at which the first excursion occurs. Another procedure is to distribute, according to the diffusion law of Markov processes, the probability mass in the phase plane of $X(\tau)$ and $\dot{X}(\tau)$ which has not passed out the safety domain for every consecutive short time interval $\Delta\tau$. Then the rate of decrease of the probability mass within the safety domain is the probability density of the first-excursion time. Crandall et al. also investigated other types of safety domains in addition to the one specified in Eq. (1). Rosenblueth and Bustamente[4] treated the total energy in a white noise excited single-degree-of-freedom system $\frac{1}{2}kX^2 + \frac{1}{2}m\dot{X}^2$ as a Markov process and computed the first-excursion probability for the total energy level passing out a safety bound. Since it is known that the total energy is not Markovian the solution is clearly an approximation.

Formulation Based upon Theory of Random Points

From a slightly different point of view, consider the fictitious situation where an excursion to or beyond a prescribed safety bound does not render the system immediately inoperative. Then it is meaningful to speak of not only the first excursion but also the second and the third excursions, etc. We are concerned with the random points along the time axis at which such excursions occur. From this viewpoint, it is quite natural to find the theory of random points developed by Stratonovich and Kuznetsov[5] applicable to the present problem.

Let us select a time interval $0 < \tau \leq t$ within which excursions occur at $\tau_1, \tau_2, \ldots, \tau_n$. The probabilistic structure of

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these random instants may be characterized by the following mathematical expectation

$$L_t[v(\tau)] = E\left\{\prod_{j=1}^{\mathfrak{N}} [1 + v(\tau_j)]\right\} \quad (2)$$

If the right-hand side of Eq. (2) is expanded and term-wise expectations are taken, we obtain

$$L_t[v(\tau)] = 1 + \sum_{s=1}^{\infty} \frac{1}{s!} \int_0^t d\tau_s \int_0^t d\tau_{s-1} \cdots \int_0^t d\tau_1 f_s(\tau_1,\tau_2,\ldots,\tau_s)v(\tau_1)\cdots v(\tau_s) \quad (3)$$

where the function $f_s(\tau_1, \tau_2, \ldots, \tau_s)$ is called the sth distribution function of the random points (following Stratonovich), and $f_s(\tau_1, \tau_2, \ldots, \tau_s)d\tau_1\, d\tau_2 \cdots d\tau_s$ gives the probability that at least one random point falls in each of the intervals $(\tau_1, \tau_1 + d\tau_1]$, $(\tau_2, \tau_2 + d\tau_2]$, $\ldots$, $(\tau_s, \tau_s + d\tau_s]$. The right-hand side of Eq. (3) is analogous to a Taylor series expansion of a moment generating function for jointly distributed random variables. In fact, L_t is known as the generating functional of the distribution functions f_s. The preceding analogy reveals that each f_s is a moment function of some sort. Indeed, if we denote the random arrival rate of the random points by $N(\tau)$ and express the total number $\mathfrak{N}(t)$ of random points within the interval $(0,t]$ as the following integral:

$$\mathfrak{N}(t) = \int_0^t N(\tau)\, d\tau \quad (4)$$

then f_s is just the sth moment function of the random process $N(\tau)$; i.e.,

$$f_s(\tau_1, \ldots, \tau_s) = E[N(\tau_1)\cdots N(\tau_s)] \quad (5)$$

In the special case where the random rate $N(\tau)$ is stationary, $f_1(\tau)$ is independent of τ, $f_2(\tau_1,\tau_2)$ is dependent only on the difference $\tau_1 - \tau_2$, etc.

It can be shown[5] that the probability of no excursion failure in $(0,t]$ may be obtained by letting $v(t) = -1$ in Eq. (3);

$$P_0(t) = 1 + \sum_{s=1}^{\infty} \frac{(-1)^s}{s!} \int_0^t \cdots \int_0^t f_s(\tau_1, \ldots, \tau_s)d\tau_1 \ldots d\tau_s \quad (6)$$

We digress to comment on the connection between Eq. (6) and a more well-known result. To do so, we impose a rather mild restriction that $P_0(t)$ approaches zero as t tends to infinity. The physical implication is that the system will fail sooner or later if it is to be operated indefinitely.† Then $F_T(t) = 1 - P_0(t)$ is the probability distribution function of the first excursion time T; i.e., $F_T(t) = \text{Prob.}\ [T \leq t]$. The probability density of the first-excursion time is obtained from a straightforward differentiation; thus

$$p_T(t) = \frac{d}{dt} F_T(t) = -\frac{d}{dt} P_0(t) = f_1(t) + \sum_{s=2}^{\infty} \frac{(-1)^{s-1}}{(s-1)!} \int_0^t \cdots \int_0^t f_s(\tau_1, \cdots, \tau_{s-1}, t)d\tau_1 \cdots d\tau_{s-1} \quad (7)$$

Eq. (7) is readily recognized to be the famous "in and exclusion" series given by S. O. Rice[6] in 1944.

An alternative expression for the probability of no excursion failure, $P_0(t)$, can be obtained from another expansion of the generating functional $L_t[v(\tau)]$;

$$L_t[v(\tau)] = \exp\left\{\sum_{s=1}^{\infty} \frac{1}{s!} \int_0^t \cdots \int_0^t g_s(\tau_1, \ldots, \tau_s) v(\tau_1)\cdots v(\tau_s)d\tau_1 \cdots d\tau_s\right\} \quad (8)$$

† This is not true, for example, for the nonstationary case considered by Shinozuka.[2]

where the g_s are called the cumulant functions‡ of the random arrival rate $N(\tau)$. These cumulant functions are related to the distribution functions f_s. Such relations can be obtained from expanding the exponential function on the right-hand side of Eq. (8) and comparing the resulting expression with Eq. (3). Thus, we obtain

$$\begin{aligned} g_1(\tau) &= f_1(\tau) \\ g_2(\tau_1,\tau_2) &= f_2(\tau_1,\tau_2) - f_1(\tau_1)f_1(\tau_2) \\ g_3(\tau_1,\tau_2,\tau_3) &= f_3(\tau_1,\tau_2,\tau_3) - f_1(\tau_1)f_2(\tau_2,\tau_3) - \\ &\quad f_1(\tau_2)f_2(\tau_1,\tau_3) - f_1(\tau_3)f_2(\tau_1,\tau_2) + \\ &\quad 2 f_1(\tau_1)f_1(\tau_2)f_1(\tau_3) \text{ etc.} \end{aligned} \quad (9)$$

The complete set of cumulant functions characterizes the correlations of arrival rates at different instants of time. In particular, if at least one of the random rates $N(\tau_1), \ldots, N(\tau_s)$ is uncorrelated with all the others, then $g_s(\tau_1, \ldots, \tau_s) = 0$.

The probability of no excursion failure can be obtained by letting $v(\tau) = -1$ in Eq. (8); thus,

$$P_0(t) = \exp\left\{\sum_{s=1}^{\infty} \frac{(-1)^s}{s!} \times \int_0^t \cdots \int_0^t g_s(\tau_1, \cdots, \tau_s)\, d\tau_1 \cdots d\tau_s\right\} \quad (10)$$

As will be seen in the next section, the computation of the moment functions f_s (thus the computation of the cumulant functions g_s) for higher s is increasingly more difficult; therefore, it is desirable to limit our consideration to lower-order statistics. The most drastic simplification is to assume that the arrival rates at different time instants constitute a system of uncorrelated random variables. For this special case, $g_s = 0$ for $s > 1$, and the probability of no arrival in the time interval $(0, t]$ is, from Eq. (10)

$$P_0(t) = \exp\left\{-\int_0^t g_1(\tau)d\tau\right\} \quad (11)$$

This probability is recognized to be that of no arrival of a Poisson system of random points (i.e., independently arriving random points).

Equation (11) has been suggested for the estimation of structural reliability.[7] However, this procedure has been criticized[8] for its underlying assumption that the excursions at different time instants are independent events, an assumption which is unrealistic for the important case of a narrow band response. Because of the slow fluctuations in the envelope of a narrow band process, once such a process passes out a safety bound there is a greater than average probability that it will cross this bound at the immediate succeeding (pseudo-) cycles. Therefore, a tendency exists for the excursions to occur in clumps,[9] and the use of Eq. (11) is conservative since it underestimates the structure reliability against a first-excursion failure.[10] However, for the trivial case where t is very small in comparison with the duration of one (pseudo-) cycle Eq. (11) is a good approximation since within such a short time interval the probability of multiple excursions can be neglected.

In passing we note that at least two schemes have been proposed to compensate the over-conservatism implicit of the assumption of Poisson excursion times. One redefines the structural failure as the first-excursion of the response envelope and regards envelope excursions as independent events;[11] another adopts a failure criterion based upon the magnitude

‡ They are called correlation functions in Stratonovich's book.[5] However, the name "correlation function" has been used most frequently for the second moment function (in this case f_2) in the U.S.A. Therefore, the alternative name is suggested here to avoid confusions.

of an envelope peak and treats different peaks as independent random variables.[12] It can be shown that the expressions for structural reliability based on these two schemes approach one another as the level of the failure threshold is raised.

It has been pointed out that the complete set of cumulant functions $g_s(\tau_1, \ldots, \tau_s)$ characterizes the correlations of the excursion rates at different times and that Eq. (11) results if we ignore the correlation of excursion rates at any two different times. More generally, substitution of the infinite series in Eq. (10) by a finite series with $1 \leq s \leq s_0$ is to disregard the correlations of excursion rates at more than s_0 different time instants. Then the probability of survival corresponding to $s_0 = 2$ is

$$P_0(t) = \exp\left\{-\int_0^t g_1(\tau)d\tau + \frac{1}{2}\int_0^t\int_0^t g_2(\tau_1,\tau_2)d\tau_1 d\tau_2\right\} \tag{12}$$

It is of interest to compare the similarities and differences between an excursion rate process $N(t)$ whose cumulant functions $g_s = 0$ for $s > 2$ and a gaussian random process. It is known that for a gaussian random process the cumulant functions of an order higher than two are zero where the first cumulant function is the mean and the second cumulant function is the covariance. However, $N(t)$ is not a gaussian random process since it cannot assume any negative values. Nor can the second cumulant $g_2(t_1,t_2)$ of $N(t)$ be regarded as a covariance function since it approaches the negative value $-g_1^2(t_1)$ as t_2 approaches t_1. It may be proper to call such an excursion rate process $N(t)$ a pseudo-gaussian process.

Nonapproaching Random Points

Instead of ignoring the higher-order cumulants of the excursion rate process completely, another reasonable scheme is to adopt suitable formulas (based on physical grounds) from which the higher statistics of the excursion rate are computed from the first two. Some of the rules to be observed are as follows. First, every $g_s(\tau_1, \ldots, \tau_s)$ must be symmetric with respect to its arguments; that is, the function remains unchanged upon an interchange of τ_j and τ_k where $1 \leq j, k \leq s$. Second, every g_s must vanish as one argument is moved away from all the others. Since a structural response process must be smooth, the probability is zero for two consecutive excursions to approach each other. The mathematical model of nonapproaching random points suggested by Stratonovich[5] satisfies these conditions. This model is defined by a recurrence formula

$$g_n(\tau_1, \ldots, \tau_n) = (-1)^{n-1}(n-1)!\; g_1(\tau_1)\ldots g_1(\tau_n)\{R(\tau_2,\tau_1)\ldots R(\tau_n,\tau_1)\}_s \tag{13}$$

where $R(\tau_j,\tau_k)$ is a symmetric function and $\{\ \}_s$ denotes a "symmetrizing operation" of the embraced quantity; that is, taking arithmetic average of all the permuted terms which are different from each other. Thus,

$$[R(\tau_2,\tau_1)]_s = R(\tau_2,\tau_1)$$

$$[R(\tau_2,\tau_1)R(\tau_3,\tau_1)]_s = 1/3[R(\tau_2,\tau_1)R(\tau_3,\tau_1) + R(\tau_3,\tau_2)R(\tau_1,\tau_2) + R(\tau_1,\tau_3)R(\tau_2,\tau_3)] \text{ etc.} \tag{14}$$

The function $R(\tau_j,\tau_k)$ is obtained by letting $n = 2$ in Eq. (13), thus,

$$R(\tau_2,\tau_1) = -[g_2(\tau_1,\tau_2)]/[g_1(\tau_1)g_1(\tau_2)] \tag{15}$$

This function bears considerable similarity to the correlation coefficient function. It attains the maximum value of unity as $\tau_1 \to \tau_2$, and vanishes as τ_1 and τ_2 are moved away from each other. We note that all the f functions are non-negative since they are moment functions of a non-negative random process, and that

$$f_2(\tau_1,\tau_2) = f_1(\tau_1)f_1(\tau_2) + g_2(\tau_1,\tau_2) = f_1(\tau_1)f_1(\tau_2)[1 - R(\tau_1,\tau_2)] \tag{16}$$

where we have used the equality $f_1 = g_1$. Therefore, the value of $R(\tau_j,\tau_k)$ is bounded above by unity. Equation (16) also shows that R tends to its maximum and f_2 tends to zero as τ_2 approaches to τ_1. Physically, this means small probabilities for two random points to be very near to each other.

When Eq. (13) is substituted into Eq. (8) the summation of the series can be evaluated explicitly; the result is

$$L_t[v(\tau)] = \exp\left\{\int_0^t \frac{\ln[1 + A(t,\tau)]}{A(t,\tau)} g_1(\tau)v(\tau)d\tau\right\} \tag{17}$$

where

$$A(t,\tau) = \int_0^t R(\tau,u)g_1(u)\, v(u)\, du.$$

Letting $v(\tau) = -1$ in Eq. (17) and denoting

$$B(t,\tau) = \int_0 R(\tau,u)\; g_1(u)\; du$$

we obtain

$$P_0(t) = L_t[-1] = \exp\left\{\int_0^t \frac{\ln[1 - B(t,\tau)]}{B(t,\tau)} g_1(\tau)\, d\tau\right\} \tag{18}$$

It should be noted, however, that although the times at which a random structural response passes outside the safety barriers do not approach each other they need not exactly obey Eq. (13). Therefore Eq. (18) is just another approximation.

It is of interest to mention an approach proposed by J. R. Rice and F. P. Beer[13] where use is also made of the first-and the second-order statistics of the excursion rate process (although in a different context). Therefore, their approach is considered to be of the same level of approximation as that of Eq. (12) or Eq. (18) in the present paper. These authors assume that successive crossings over a given level are renewal events. This renewal approximation was applied to two gaussian random processes, one being a truncated white noise, the other having a constant spectral density over a 1-octave band. It was found that the renewal approximation was the least appropriate for the second case; therefore, it would be even less appropriate for a typical structural response process having an effective band-width much smaller than one octave.

Statistics of Excursion Rate

Application of the theory of random points to the first excursion failure problem requires a knowledge of the statistics of the excursion rate $N(\tau)$. On the level of a second-order analysis we require specifically $f_1(\tau) = E[N(\tau)]$ and $f_2(\tau_1,\tau_2) = E[N(\tau_1)\; N(\tau_2)]$. We shall derive the necessary formulas for these mathematical expectations using a technique essentially due to Middleton.[14]

In Fig. 1 we depict the corresponding sample functions of three interrelated random processes. The first, $x(\tau)$, is a typical sample function of a random structural response $X(\tau)$; the second, $y(\tau)$, is that of

$$Y(\tau) = \mathbf{1}[X(\tau) - b] + \mathbf{1}[-X(\tau) - a] \tag{19}$$

and third, $\dot{y}(\tau)$, is that of

$$\dot{Y}(\tau) = (d/dt)Y(\tau) = \dot{X}(\tau)\delta[X(\tau) - b] - \dot{X}(\tau)\delta[-X(\tau) - a] \tag{20}$$

where $-a$ and b are the excursion barriers, $\mathbf{1}[\;\;]$ denotes a Heaviside step function, and $\delta[\;\;]$ represents a Dirac delta function. Note that all the impulses in $\dot{y}(\tau)$ are unit impulses with alternate signs. It is when an impulse is positive that the response $x(\tau)$ passes out the safety domain $(-a,b)$. Let $\mathfrak{N}(t]$ be the total number of such excursions in the time interval $(0, t)$, and recall Eq. (4).

Then it is clear that $N(\tau)$ is just the positive part of $\dot{Y}(\tau)$; that is,

$$N(\tau) = \dot{X}(\tau)\delta[X(\tau) - b]\mathbf{1}[\dot{X}(\tau)] - \dot{X}(\tau)\delta[-X(\tau) - a]\mathbf{1}[-\dot{X}(\tau)] \quad (21)$$

Equation (21) shows that the moments (of various orders) of $N(\tau)$ can be computed from the joint distribution (of various orders) of $X(\tau)$ and $\dot{X}(\tau)$. In particular,

$$f_1(\tau) = E[N(\tau)] = \int_0^\infty \dot{x}p_{X\dot{X}}(b,\dot{x},\tau)d\dot{x} - \int_{-\infty}^0 \dot{x}p_{X\dot{X}}(-a,\dot{x},\tau)d\dot{x} \quad (22)$$

and

$$f_2(\tau_1,\tau_2) = E[N(\tau_1)N(\tau_2)] = \int_0^\infty d\dot{x}_1 \int_0^\infty \times \dot{x}_1\dot{x}_2 p_{X\dot{X}}(b,\dot{x}_1\tau_1;\; b,\dot{x}_2,\tau_2)d\dot{x}_2 + \int_{-\infty}^0 d\dot{x}_1 \int_{-\infty}^0 \dot{x}_1\dot{x}_2 p_{X\dot{X}}(-a,\dot{x}_1,\tau_1;\; -a,\dot{x}_2,\tau_2)d\dot{x}_2 - \int_0^\infty d\dot{x}_1 \int_{-\infty}^0 \dot{x}_1\dot{x}_2 p_{X\dot{X}}(b,\dot{x}_1,\tau_1;\; -a,\dot{x}_2\tau_2)d\dot{x}_2 - \int_{-\infty}^0 d\dot{x}_1 \int_0^\infty \dot{x}_1\dot{x}_2 p_{X\dot{X}}(-a,\dot{x}_1,\tau_1;\; b,\dot{x}_2,\tau_2)d\dot{x}_2 \quad (23)$$

where the $p_{X\dot{X}}$ are the joint probability densities of $X(\tau)$ and $\dot{X}(\tau)$, and they have the following physical meanings:

$$p_{X\dot{X}}(x,\dot{x},\tau)\,dx\,d\dot{x} = \text{Prob}[\{x \le X(\tau) \le x + dx\} \cap \{\dot{x} \le \dot{X}(\tau) \le \dot{x} + d\dot{x}\}]$$

$$p_{X\dot{X}}(x_1,\dot{x}_1,\tau_1;\; x_2,\dot{x}_2,\tau_2)dx_1 d\dot{x}_1 dx_2 d\dot{x}_2 = \text{Prob}[\{x_1 \le X(\tau_1) \le x_1 + dx_1\} \cap \{\dot{x}_1 \le \dot{X}(\tau_1) \le \dot{x}_1 + d\dot{x}_1\} \cap \{x_2 \le X(\tau_2) \le x_2 + dx_2\} \cap \{\dot{x}_2 \le \dot{X}(\tau_2) \le \dot{x}_2 + d\dot{x}_2\}]$$

Single-Degree-of-Freedom Oscillator

To proceed further with the computation of the statistics of the excursion rate, we must specify the various joint probability densities of $X(\tau)$ and $\dot{X}(\tau)$. One case of special importance is where $X(\tau)$ is a stationary response of a single-degree-of-freedom oscillator under the excitation of a gaussian white noise. This case will be chosen as an example for further discussions. For simplicity, it also will be assumed that the excursion bounds are symmetrical (i.e., $a = b$). We now have, for use in Eq. (22)

$$p_{X\dot{X}}(x,\dot{x}) = (2\pi\omega_0\sigma^2)^{-1} \exp\left(-\frac{x^2}{2\sigma^2} - \frac{\dot{x}_2}{2\omega_0{}^2\sigma^2}\right) \quad (24)$$

where ω_0 is the natural frequency of the oscillator and σ and $\omega_o\sigma$ are the standard deviations of $X(\tau)$ and $\dot{X}(\tau)$, respectively. Since this probability density is independent of time, the argument τ has been dropped from the left-hand side of Eq. (24). Substituting (24) into (22) and recasting the result in nondimensional form we obtain

$$f_1/\omega_0 = (1/\pi)\exp-[a^2/2\sigma^2] \quad (25)$$

The joint probability density required in Eq. (23) for the correlation function of $N(\tau)$ is of the form of a four-dimensional gaussian distribution. For a stationary $X(\tau)$ this probability density is dependent only on the time difference

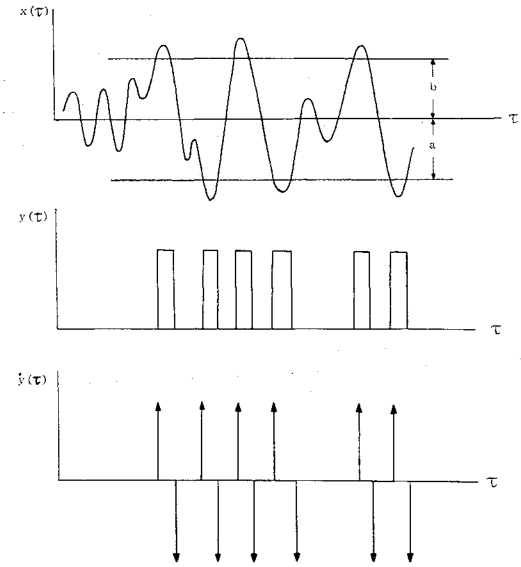

Fig. 1 Corresponding sample functions of three interrelated random processes.

$\tau_1 - \tau_2$. It is convenient to re-name the variables as follows: $u = \omega_0(\tau_2 - \tau_1)$, $z_1 = x_1$, $z_2 = \dot{x}_1$, $z_3 = \dot{x}_2$, $z_4 = x_2$. Then

$$p_{X\dot{X}}(x_1,\dot{x}_1,\tau_1;\; x_2,\dot{x}_2,\tau_2) = p(z_1,z_2,z_3,z_4,u) = (2\pi)^{-2}\Delta^{-1/2}\exp\left(-\frac{1}{2}\sum_{j=1}^{4}\sum_{k=1}^{4}\alpha_{jk}z_j z_k\right) \quad (26)$$

The essential components of this joint probability density Δ and α_{jk} can be expressed in terms of the correlation coefficient function of the structural response

$$\rho(u) = [\cos(1 - \zeta^2)^{1/2}u + \zeta(1 - \zeta^2)^{-1/2} \times \sin(1 - \zeta^2)^{1/2}|u|]\exp(-\zeta|u|) \quad (27)$$

where ζ is the ratio of damping to the critical damping. Specifically, Δ is the determinant of the following matrix of variances and covariances:

$$\sigma^2\begin{bmatrix} \rho(0) & 0 & \omega_0\rho'(u) & \rho(u) \\ 0 & -\omega_0{}^2\rho''(0) & -\omega_0{}^2\rho''(u) & -\omega_0\rho'(u) \\ \omega_0\rho'(u) & -\omega_0{}^2\rho''(u) & -\omega_0{}^2\rho''(0) & 0 \\ \rho(u) & -\omega_0\rho'(u) & 0 & \rho(0) \end{bmatrix} \quad (28)$$

Thus $\Delta = \sigma^8\omega_0{}^4D(u)$ with

$$D(u) = \{[\rho'(u)]^2 - 1 - \rho(u)\rho''(u)\}^2 - [\rho(u) + \rho''(u)]^2 \quad (29)$$

The α_{jk} are the elements of the inverse of the matrix in (28). Since this matrix is symmetrical with respect to both the diagonal and the cross-diagonal, so also is its inverse. Therefore, it is only necessary to specify the following six elements:

$$\alpha_{11} = \sigma^{-2}\beta_{11} = \sigma^{-2}D^{-1}(u)\{1 - [\rho''(u)]^2 + \rho''(u)[\rho'(u)]^2\}$$
$$\alpha_{22} = \sigma^{-2}\omega_0{}^{-2}\beta_{22} = \sigma^{-2}\omega_0{}^{-2}D^{-1}(u)\{1 - \rho^2(u) - [\rho'(u)]^2\}$$
$$\alpha_{12} = \sigma^{-2}\omega_0{}^{-1}\beta_{12} = -\sigma^{-2}\omega_0{}^{-1}D^{-1}(u)\rho'(u)[\rho(u) + \rho''(u)]$$
$$\alpha_{13} = \sigma^{-2}\omega_0{}^{-1}\beta_{13} = \sigma^{-2}\omega_0{}^{-1}D^{-1}(u)\rho'(u)\{[\rho'(u)]^2 - \rho(u)\rho''(u) - 1\}$$
$$\alpha_{14} = \sigma^{-2}\beta_{14} = \sigma^{-2}D^{-1}(u)\{\rho(u)[\rho''(u)]^2 - \rho(u) - [\rho'(u)]^2\rho''(u)\}$$
$$\alpha_{23} = \sigma^{-2}\omega_0{}^{-2}\beta_{23} = \sigma^{-2}\omega_0{}^{-2}D^{-1}(u)\{\rho''(u)[1 - \rho^2(u)] + \rho(u)[\rho'(u)]^2\} \quad (30)$$

For the present example, the first two double integrals in Eq. (23) are equal and the last two double integrals are also

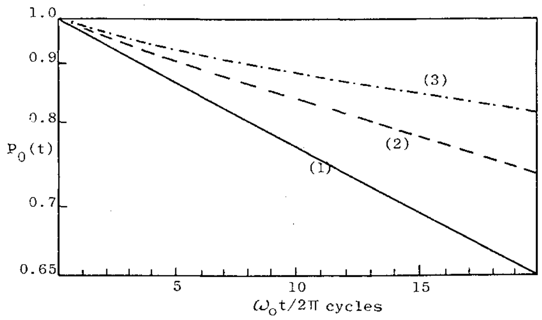

Fig. 2 Probability of survival for simple oscillator with $\zeta = 0.05$ and failure bounds $\pm 3\sigma$. 1) Poisson excursions, 2) nonapproaching excursions, 3) excursions with pseudo-gaussian rate.

equal. These integrals have the general form

$$I = \iint z_2 z_3 \exp[-c_1(z_2^2 + z_3^2) - c_2 z_2 - c_3 z_3 - c_4 z_2 z_3] dz_2 dz_3 \quad (31)$$

An alternative form which is somewhat easier to evaluate is as follows:

$$I = -\frac{\partial}{\partial c_4} \iint \exp[-c_1(z_2^2 + z_3^2) - c_2 z_2 - c_3 z_3 - c_3 z_2 z_3] dz_2 dz_3 \quad (32)$$

After a considerable amount of algebraic work, we obtain

$$\bar{f}_2(u) = \frac{f_2(\tau_1,\tau_2)}{g_1^2} = C_1(u) \int_0^{\pi/4} \frac{\sin 2\theta}{(\beta_{22} + \beta_{23}\sin 2\theta)^2} \left[1 + w_1^2 + \pi^{1/2} w_1\left(\frac{3}{2} + w_1^2\right)(\operatorname{erf} w_1)\exp(w_1^2)\right] d\theta + C_2(u) \int_0^{\pi/4} \frac{\sin 2\theta}{(\beta_{22} - \beta_{23}\sin 2\theta)^2} \left[1 + w_2^2 + \pi^{1/2} w_2\left(\frac{3}{2} + w_2^2\right)(\operatorname{erf} w_2)\exp(w_2^2)\right] d\theta \quad (33)$$

where

$$C_1(u) = D^{-1/2} \exp\left[-\left(\frac{a}{\sigma}\right)^2 (\beta_{11} + \beta_{14} - 1)\right]$$

$$C_2(u) = D^{-1/2} \exp\left[-\left(\frac{a}{\sigma}\right)^2 (\beta_{11} - \beta_{14} - 1)\right]$$

$$w_1 = \left(\frac{a}{\sigma}\right) \frac{(\beta_{12} - \beta_{13})(\cos\theta - \sin\theta)}{[2(\beta_{22} + \beta_{23}\sin 2\theta)]^{1/2}}$$

$$w_2 = \left(\frac{a}{\sigma}\right) \frac{(\beta_{12} + \beta_{13})(\cos\theta - \sin\theta)}{[2(\beta_{22} - \beta_{23}\sin 2\theta)]^{1/2}}$$

Further simplification of Eq. (33) appears unlikely, and the integration on θ in this equation must be carried out numerically on a digital computer. It is interesting to note that integrals of the same type have been obtained in Refs. 6 and 13.

Numerical Results

The estimates for the probability of survival was computed for a single-degree-of-freedom linear oscillator based upon three different criteria: 1) Poisson excursions, 2) the nonapproaching excursions, and 3) the pseudo-gaussian excursions. Typical results for the case of stationary response are plotted in Figs. 2–5. It was found that, except at t values in the order of one natural cycle or less where all three estimates nearly coincide, the pseudo-gaussian estimate is usually the highest and the Poisson estimate is the lowest. This order may be changed in some time regions in the case of low thresholds, say about 1.5 σ, where again all three estimates nearly coincide (not plotted).

An indication of the relative merit of the preceding three estimates is provided by the numerical result of Crandall, Chandiramani, and Cook[3] (their Fig. 11 converted to the survival probability) which is also included in Fig. 5. Their results are expected to be accurate for the particular dynamical system since it was obtained from direct simulation of a large number of samples or from a numerical diffusion procedure with fine grid of cells on the phase plane. Although this information is available only for the particular system and for a time up to five natural cycles, it clearly indicates that the Poisson estimate is far too conservative and that the two second-order estimates are considerably better. Such a comparison should provide some confidence in the second-order methods proposed in this paper. However, our objective is, of course, to provide a general scheme which is also applicable to other dynamical systems, including multi-degree-of-freedom systems, and for longer time intervals where simulation results such as that of Ref. 3 are unavailable.

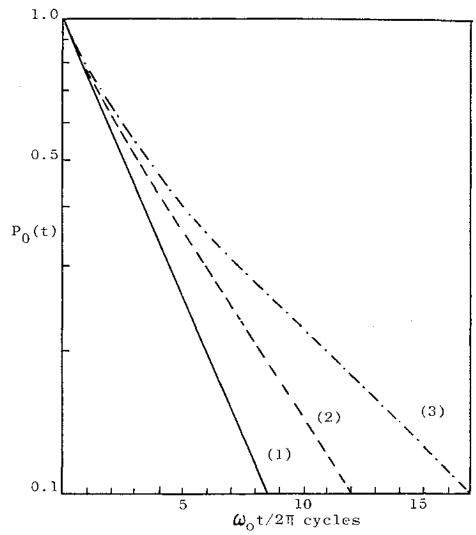

Fig. 3 Probability of survival for simple oscillator with $\zeta = 0.05$ and failure bounds $\pm 2\sigma$. 1) Poisson excursions, 2) nonapproaching excursions, 3) excursions with pseudo-gaussian rate.

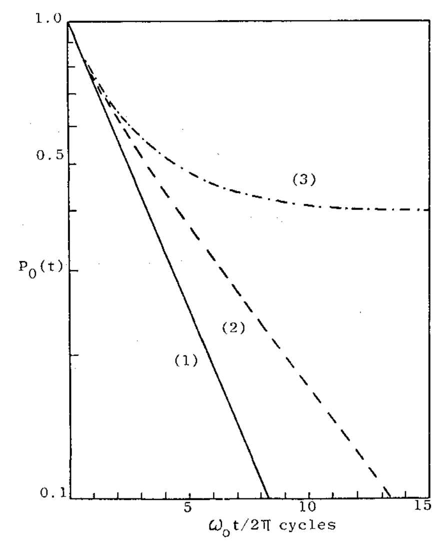

Fig. 4 Probability of survival for simple oscillator with $\zeta = 0.03$ and failure bounds $\pm 2\sigma$. 1) Poisson excursions, 2) nonapproaching excursions, 3) excursions with pseudo-gaussian rate.

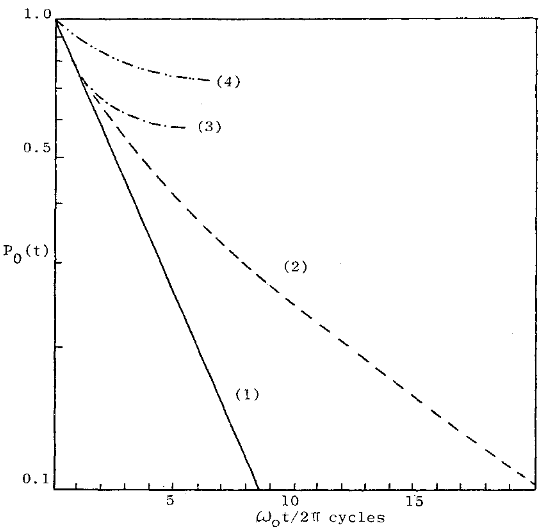

Fig. 5 Probability of survival for simple oscillator with $\zeta = 0.01$ and failure bounds $\pm 2\sigma$. 1) Poisson excursions, 2) nonapproaching excursions, 2) excursions with pseudo-gaussian rate, 4) result from Ref. 3.

For all the cases computed, the nonapproaching excursion estimate appears to be more conservative than the pseudo-gaussian estimate, but it is always a meaningful estimate regardless of the magnitude of the failure threshold, the value of damping ratio, and the length of time t. On the other hand, when damping is very light the pseudo-gaussian estimate is not valid in some time regions. It has been found, for example, that for $\zeta = 0.01$ and threshold bounds $\pm 2\sigma$ the pseudo-gaussian estimate is valid only up to four natural cycles, beyond which time this estimate fluctuates slightly and then even increases with time. Since a probability of survival must be nonincreasing with time, the invalid portion of the estimate must be rejected. The abnormal behavior in some regions of a pseudo-gaussian estimate may be due to the fact that by imposing $g_s = 0$ for $s > 2$ we have forced upon the excursion rate process $N(\tau)$ with certain gaussian properties including the possibility of "negative" rate, consequently the possibility of increasing probability of survival with time. Thus, for the case of very light damping, we must either choose a nonapproaching excursion estimate or use a truncated series with nonzero higher-order g functions. It was found that for $\zeta = 0.05$ the pseudo-gaussian estimate was satisfactory throughout the entire time range of 20 cycles investigated in this study.

We remark that the pseudo-gaussian estimate is not the only one which shows inconsistencies in some time regions. For example, the renewal approach proposed in Ref. 13 also results in negative recurrence probability density in certain cases. It is felt that such abnormal behavior may result from the omission of higher-order statistics in the analysis (although the nonapproaching excursion estimate has been found to be free from inconsistency throughout the positive time domain). We further remark that the general scheme proposed in this paper applies to both stationary and nonstationary response, although only the stationary examples have been presented herein.

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